Optimizing the AC Resistance of Multilayer Transformer Windings with Arbitrary Current Waveforms[1]

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Abstract – AC losses due to non-sinusoidal current waveforms have been found by calculating the losses at harmonic frequencies when the Fourier coefficients are known. An optimized foil or layer thickness in a winding may be found by applying the Fourier analysis over a range of thickness values. This paper presents a new formula for the optimum foil or layer thickness, without the need for Fourier coefficients and calculations at harmonic frequencies. The new formula requires the rms value of the current waveform and the rms value of its derivative. It is simple, straightforward and applies to any periodic waveshape.

NOMENCLATURE

d	Thickness of foil or layer.
D	Duty cycle.
f	Fundamental frequency of current waveform in Hz.
I_{dc}	Average value of current.
I_n	rms value of the n^{th} harmonic.
I_{rms}	rms value of the current waveform.
I'_{rms}	rms value of the derivative of the current waveform.
k_{p_n}	Ratio of the ac resistance to dc resistance at n^{th} harmonic frequency.
N	Number of turns per layer.
n	Harmonic number.
p	Number of layers.
r_o	Radius of bare wire in wire-wound winding.
R_{ac}	ac resistance of a winding with sinusoidal excitation.
R_{dc}	dc resistance of a winding.
R_{eff}	Effective ac resistance of a winding, with arbitrary current waveform.
R_δ	dc resistance of a winding of thickness δ_0.
t_r	Rise time (0-100%).
T	Period of the current waveform.
δ_0	$\sqrt{\dfrac{2}{\omega\mu_0\sigma}}$. Skin depth at fundamental frequency, $\omega=2\pi f$.
δ_n	Skin depth at the n^{th} harmonic frequency.
Δ	d/δ_0.
η	Porosity factor, see Fig. 1.

I. INTRODUCTION

Transformers are operated at high frequencies in order to reduce their size [1]. Switching circuits and resonant circuits have greatly improved the efficiencies of power supplies. These power supplies have non-sinusoidal current waveforms and give rise to additional ac losses due to harmonics. AC resistance effects due to sinusoidal currents were treated by Bennett and Larson [2] and this work was tailored specifically for transformers by Dowell [3]. These works are based on a one-dimensional solution of the diffusion equation as applied to conducting parallel plates. Dowell's formula has been found to reliably predict the increased resistance in cylindrical windings where the foil or layer thickness is less than 10% of the radius of curvature. The formula has been utilized in many applications such as planar magnetics by Kassakian [4] and Sullivan [5], matrix transformers by Williams [6], toroidal inductors by Cheng [7], distributed air-gaps by Evans [8] and slot bound conductors by Hanselman [9].

With the advent of switch mode power supplies, attention switched to non-sinusoidal current waveforms. These currents were decomposed into Fourier components; the harmonic components are orthogonal so that the total loss is equal to the sum of the losses calculated by Dowell's formula for the amplitude and frequency of each harmonic in turn. Venkatraman [10] showed that for a pulsed waveform typical of a forward converter, there is an optimum layer thickness to minimize ac losses. Carsten [11] extended the analysis to square waveforms, which are encountered in full bridge

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converters and to triangular waveforms, which occur in filter chokes. Vandelac [12] extended the analysis to flyback converters. The optimum layer thickness is found as follows:

- Calculate the Fourier coefficients
- Calculate the losses at each harmonic frequency
- Calculate the total losses for each thickness in a range of values
- Read the optimum thickness from a graph of ac resistance versus layer thickness

Typically this might involve loss calculations at up to 30 harmonics for up to 10 thickness values in order to find the optimum value. Furthermore Fourier coefficients are only available for a few waveforms.

This paper presents a new formula for ac resistance and the optimum layer thickness for any current waveform. The formula only requires knowledge of the rms value of the current waveform and the rms value of its derivative. Both these quantities can be easily measured or calculated with simulation programs such as PSPICE. The results are just as accurate as the cumbersome method based on Fourier analysis.

II. THE AC RESISTANCE

The real part of Dowell's formula gives the ac to dc resistance factor:

$$\frac{R_{ac}}{R_{dc}} = \Delta\left[\frac{\sinh(2\Delta)+\sin(2\Delta)}{\cosh(2\Delta)-\cos(2\Delta)} + \frac{2(p^2-1)}{3}\frac{\sinh(\Delta)-\sin(\Delta)}{\cosh(\Delta)+\cos(\Delta)}\right] \quad (1)$$

where Δ is the ratio of the layer thickness d to the skin depth δ_0. This is a very good approximation to the original cylindrical solution, particularly if the layer thickness is less than 10% of the radius of curvature. Windings which consist of round conductors, or foils which do not extend the full winding window, may be treated as foils with equivalent thickness d and effective conductivity $\sigma_w = \eta\sigma$. This calculation is shown graphically in Fig. 1, a detailed treatment of wire conductors is given by Ferreira [13] and Jongsma [14]. The orthogonality of skin and proximity effects in wire windings is described by Ferreira [13].

The trigonometric and hyperbolic functions in (1) may be represented by the series expansions:

$$\frac{\sinh 2\Delta+\sin 2\Delta}{\cosh 2\Delta-\cos 2\Delta} \approx \frac{1}{\Delta}+\frac{4}{45}\Delta^3-\frac{16}{4725}\Delta^7+O(\Delta^{11}), \quad (2)$$

$$\frac{\sinh\Delta-\sin\Delta}{\cosh\Delta+\cos\Delta} \approx \frac{1}{6}\Delta^3-\frac{17}{2520}\Delta^7+O(\Delta^{11}). \quad (3)$$

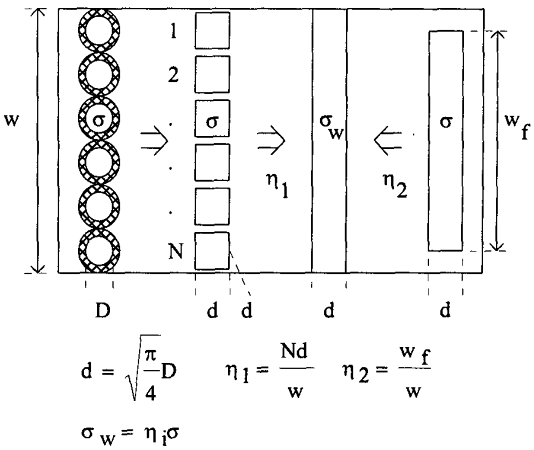

Fig. 1. Porosity factor for foils and round conductors.

If only terms up to the order of Δ^3 are used, the relative error incurred in (2) is less than 1.2% for $\Delta<1.2$ and the relative error in (3) is less than 4.1% for $\Delta<1$ and is less than 8.4% if $\Delta<1.2$. The asymptotic values of the functions on the left hand side of (2) and (3) are 1 for $\Delta>2.5$. Terms up to the order of Δ^3 are sufficiently accurate to account for the Fourier harmonics which are used to predict the optimum value of Δ which is normally in the range 0.3-1.

Thus (1) becomes

$$\frac{R_{ac}}{R_{dc}} = 1+\frac{\Psi}{3}\Delta^4 \quad (4)$$

where

$$\psi = \frac{5p^2-1}{15}. \quad (5)$$

An arbitrary periodic current waveform, may be represented by its Fourier Series

$$i(t) = I_{dc} + \sum_{n=1}^{\infty} a_n \cos n\omega t + b_n \sin n\omega t. \quad (6)$$

The sine and cosine terms may be combined to give an alternative form

$$i(t) = I_{dc} + \sum_{n=1}^{\infty} c_n \cos(n\omega t+\varphi_n) \quad (7)$$

where I_{dc} is the dc value of i(t) and c_n is the amplitude of the n^{th} harmonic with corresponding phase φ_n. The rms value of the n^{th} harmonic is $I_n = c_n/\sqrt{2}$.

The total power loss due to all the harmonics is

$$P = R_{dc} I_{dc}^2 + R_{dc} \sum_{n=1}^{\infty} k_{p_n} I_n^2 \tag{8}$$

where k_{p_n} is the ac resistance factor at the n^{th} harmonic frequency, which may be found from (1)

$$k_{p_n} = \sqrt{n}\Delta \left[\frac{\mathrm{Sinh}(2\sqrt{n}\Delta) + \mathrm{Sin}(2\sqrt{n}\Delta)}{\mathrm{Cosh}(2\sqrt{n}\Delta) - \mathrm{Cos}(2\sqrt{n}\Delta)} + \frac{2(p^2-1)}{3} \frac{\mathrm{Sinh}(\sqrt{n}\Delta) - \mathrm{Sin}(\sqrt{n}\Delta)}{\mathrm{Cosh}(\sqrt{n}\Delta) + \mathrm{Cos}(\sqrt{n}\Delta)} \right]. \tag{9}$$

R_{eff} is the ac resistance due to i(t) so that $P=R_{eff}I_{rms}^2$, I_{rms} being the rms value of i(t). Thus the ratio of effective ac resistance to dc resistance is

$$\frac{R_{eff}}{R_{dc}} = \frac{I_{dc}^2 + \sum_{n=1}^{\infty} k_{p_n} I_n^2}{I_{rms}^2}. \tag{10}$$

The skin depth at the n^{th} harmonic is $\delta_n = \delta_0/\sqrt{n}$ and, from (4), the ac resistance factor at the n^{th} harmonic frequency is

$$k_{p_n} = 1 + \frac{\Psi}{3} n^2 \Delta^4. \tag{11}$$

Substituting (11) into (10) yields

$$\frac{R_{eff}}{R_{dc}} = \frac{I_{dc}^2 + \sum_{n=1}^{\infty} I_n^2 + \frac{\Psi}{3}\Delta^4 \sum_{n=1}^{\infty} n^2 I_n^2}{I_{rms}^2}. \tag{12}$$

The rms value of the current in terms of its harmonics is

$$I_{rms}^2 = I_{dc}^2 + \sum_{n=1}^{\infty} I_n^2. \tag{13}$$

The derivative of i(t) in (7) is

$$\frac{di}{dt} = -\omega \sum_{n=1}^{\infty} n c_n \sin(n\omega t + \phi_n) \tag{14}$$

and the rms value of the derivative of the current is [15]

$$I'^2_{rms} = \omega^2 \sum_{n=1}^{\infty} \frac{n^2 c_n^2}{2} = \omega^2 \sum_{n=1}^{\infty} n^2 I_n^2 \tag{15}$$

which, upon substitution into (12) using (13), yields

$$\frac{R_{eff}}{R_{dc}} = 1 + \frac{\Psi}{3} \Delta^4 \left[\frac{I'_{rms}}{\omega I_{rms}} \right]^2. \tag{16}$$

This is a straightforward expression for the effective resistance of a winding with an arbitrary current waveform and it may be evaluated without knowledge of the Fourier coefficients of the waveform.

III. The Optimum Conditions

There is an optimum value of d, which gives a minimum value of effective ac resistance. Define R_δ as the resistance of a foil of thickness δ_0 such that

$$\frac{R_\delta}{R_{dc}} = \frac{d}{\delta_0} = \Delta \tag{17}$$

which implies that

$$\frac{R_{eff}}{R_{dc}} = \Delta \frac{R_{eff}}{R_\delta}. \tag{18}$$

Evidently, a plot of R_{eff} / R_δ versus Δ has the same shape as a plot of R_{eff} versus d at a given frequency. A 3-D plot of R_{eff} / R_δ versus Δ with p, the number of layers in the winding, on the third axis is shown in Fig. 2.

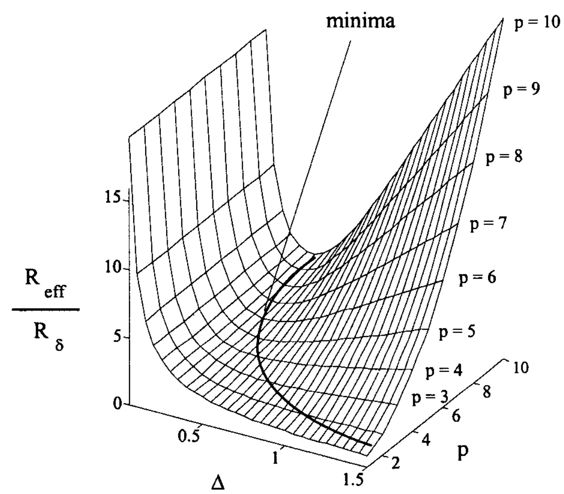

Fig. 2. Plot of ac resistance versus Δ and number of layers p.

For each value of p there is an optimum value of Δ where the ac resistance of the winding is minimum. These optimum points lie on the line marked minima in the graph and the corresponding value of the optimum layer thickness is

$$d_{opt} = \Delta_{opt}\delta_0. \tag{19}$$

From (18), using (16)

$$\frac{R_{eff}}{R_\delta} = \frac{1}{\Delta} + \frac{\Psi}{3}\Delta^3 \left[\frac{I'_{rms}}{\omega I_{rms}} \right]^2 \tag{20}$$

The optimum value of Δ is found by taking the derivative of (20) and setting it to zero.

$$\frac{d}{d\Delta}\left(\frac{R_{eff}}{R_\delta}\right)=-\frac{1}{\Delta^2}+\psi\Delta^2\left[\frac{I'_{rms}}{\omega I_{rms}}\right]^2=0. \quad (21)$$

The optimum value of Δ is

$$\Delta_{opt}=\frac{1}{\sqrt[4]{\psi}}\sqrt{\frac{\omega I_{rms}}{I'_{rms}}}. \quad (22)$$

Substituting this result into (16) yields the optimum value of the effective ac resistance with an arbitrary periodic current waveform:

$$\left(\frac{R_{eff}}{R_{dc}}\right)_{opt}=\frac{4}{3}. \quad (23)$$

Jongsma [14] and Snelling [16] have already established this result for sinusoidal excitation. The corresponding value for wire conductors with sinusoidal excitation is 3/2 [14,16].

We may also write (16) in term of Δ_{opt}

$$\frac{R_{eff}}{R_{dc}}=1+\frac{1}{3}\left(\frac{\Delta}{\Delta_{opt}}\right)^4. \quad (24)$$

We now have a set of simple formulas with which to find the optimum value of the foil or layer thickness of a winding and its effective ac resistance, these formulas are based on the rms value of the current waveform and the rms value of its derivative.

IV. VALIDATION

Consider the pulsed current waveform in Fig. 3 along with its derivative, which is typical of a forward converter. This waveform has a Fourier series:

$$i(t)=I_o\left(\frac{1}{2}-\frac{t_r}{T}\right)+\sum_{n=1}^{\infty}\frac{4I_o}{n^3\pi^3\left(\frac{t_r}{T}\right)^2}\times \left[1-\cos\left(\frac{n\pi t_r}{T}\right)\right]\sin\left(n\pi\left(\frac{1}{2}-\frac{t_r}{T}\right)\right)\cos\left(n\left(\omega t-\frac{\pi}{2}\right)\right) \quad (25)$$

The rms value of I(t) and the the rms value of its derivative are

$$I_{rms}=I_o\sqrt{0.5-\frac{37t_r}{30T}}$$
$$I'_{rms}=I_o\sqrt{\frac{8}{3t_rT}} \quad (26)$$

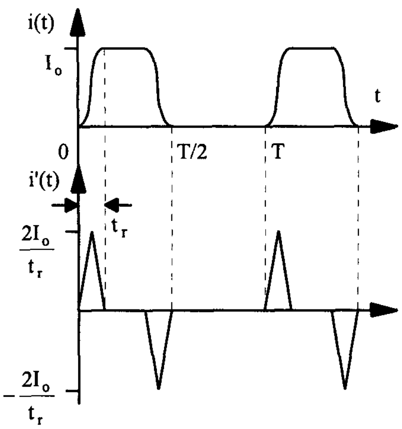

Fig.3. Pulsed current waveform and its derivative.

The optimum value of Δ given by the Fourier series (25), for p=6 and t_r/T=4% is 0.418 and the value given by the proposed formula (22) is 0.387 which represents an error of 7.4%.

Waveform 5 in Table II is an approximation to the pulse in Fig. 3 and the optimum value of Δ using Fourier analysis is 0.425 which represents an error of 7.2% when compared to the Fourier analysis of the waveform given by (25). Evidently waveforms with known Fourier series are often approximations to the actual waveform and can give rise to errors which are of the same order as the new formula, which is simpler to evaluate.

At 50 kHz the skin depth in copper is 0.295 mm. With Δ_{opt}=0.418, d_{opt}=0.295x0.418=0.123 mm.

The new formula may be validated by comparing the value of Δ_{opt} obtained with (22) and the value obtained with Fourier analysis by plotting (10), using (9), over a range of values of Δ and finding the optimum value. The results are shown in Table I for the waveforms in Table II. In general the agreement is within 3%, with the exception of waveform 5 where the error is 6.5%. For the Fourier analysis 19 harmonics were evaluated and R_{eff}/R_δ was calculated for 20 values of Δ. This means that (9) was computed 380 times for each waveform in order to find the optimum layer thickness, (22) was computed once for the same result. For 1 to 3 layers the accuracy of the proposed formula is not very good, however, as evidenced by Fig. 2, the plot of R_{eff}/R_δ is almost flat in the region of the optimum value of Δ, and therefore the error in the ac resistance is negligible.

TABLE I
VALIDATION RESULTS,
p=6, D=0.4, t_r/T=4%

Waveform No.	Fourier Analysis	New Formula
1	0.539	0.538
2	0.490	0.481
3	0.348	0.340
4	0.429	0.415
5	0.416	0.389
6	0.328	0.314
7	0.515	0.507
8	0.469	0.458
9	0.333	0.324

V. CONCLUSIONS

A new formula has been presented to find the optimum foil or layer thickness in a multilayer winding. The formula applies to any arbitrary periodic current waveform. It is computationally easier to use than Fourier analysis while enjoying the same level of accuracy. It has a wider range of application than the Fourier approach by virtue of its simplicity.

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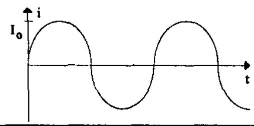
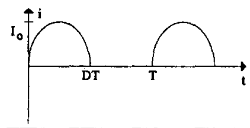
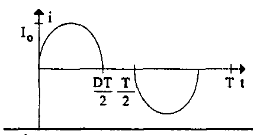
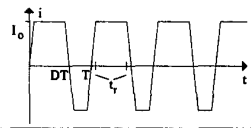
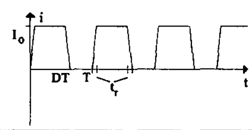
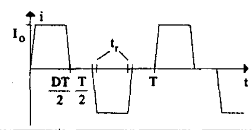
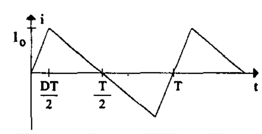
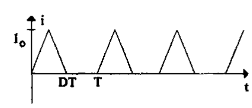
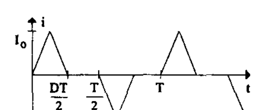
TABLE II
FORMULAS FOR THE OPTIMUM THICKNESS OF A WINDING FOR VARIOUS WAVEFORMS, $\Psi=(5p^2-1)/15$, p=NO. OF LAYERS

Current Waveform	I_{rms} and I_{rms}'	Fourier Series, i(t)	Δ_{opt}
1. 	$I_{rms}=\frac{I_o}{\sqrt{2}}$ $I_{rms}'=\frac{2\pi}{T\sqrt{2}}I_o$	$\text{Sin}(\omega t)$	$\Delta_{opt}=\sqrt[4]{\frac{1}{\Psi}}$
2. 	$I_{rms}=I_o\sqrt{\frac{D}{2}}$ $I_{rms}'=I_o\frac{\pi}{DT}\sqrt{\frac{D}{2}}$	$\frac{2DI_o}{\pi}+\sum_{n=1}^{\infty}\frac{4DI_o}{\pi}\left\{\frac{\text{Cos}(n\pi D)}{(1-4n^2D^2)}\right\}\text{Cos}(n\omega t)$ ♦	$\Delta_{opt}=\sqrt[4]{\frac{4D^2}{\Psi}}$
3. 	$I_{rms}=I_o\sqrt{\frac{D}{2}}$ $I_{rms}'=I_o\frac{2\pi}{DT}\sqrt{\frac{D}{2}}$	$\sum_{n=1,\,odd}^{\infty}\frac{4DI_o}{\pi}\left\{\frac{\text{Cos}(n\pi D/2)}{(1-n^2D^2)}\right\}\text{Cos}(n\omega t)$ ♦	$\Delta_{opt}=\sqrt[4]{\frac{D^2}{\Psi}}$
4. 	$I_{rms}=I_o\sqrt{1-\frac{8t_r}{3T}}$ $I_{rms}'=I_o\sqrt{\frac{4}{t_rT}}$	$I_o(2D-1)+\sum_{n=1}^{\infty}\frac{4I_o}{n\pi}\text{Sin}(n\pi D)$ $\times\text{Sinc}\left(n\pi\frac{2t_r}{T}\right)\text{Cos}(n\omega t)$	$\Delta_{opt}=\sqrt[4]{\frac{\left[1-\frac{8t_r}{3T}\right]\pi^2\frac{t_r}{T}}{\Psi}}$
5. 	$I_{rms}=I_o\sqrt{D-\frac{4t_r}{3T}}$ $I_{rms}'=I_o\sqrt{\frac{2}{t_rT}}$	$I_o\left(D-\frac{t_r}{T}\right)+\sum_{n=1}^{\infty}\frac{2I_o}{n\pi}\text{Sin}\left(n\pi\left(D-\frac{t_r}{T}\right)\right)$ $\times\text{Sinc}\left(n\pi\frac{t_r}{T}\right)\text{Cos}(n\omega t)$	$\Delta_{opt}=\sqrt[4]{\frac{\left[D-\frac{4t_r}{3T}\right]2\pi^2\frac{t_r}{T}}{\Psi}}$
6. 	$I_{rms}=I_o\sqrt{D-\frac{8t_r}{3T}}$ $I_{rms}'=I_o\sqrt{\frac{4}{t_rT}}$	$\sum_{n=1,\,odd}^{\infty}\frac{4I_o}{n\pi}\text{Sin}\left(n\pi\left(\frac{D}{2}-\frac{t_r}{T}\right)\right)$ $\times\text{Sinc}\left(n\pi\frac{t_r}{T}\right)\text{Cos}(n\omega t)$	$\Delta_{opt}=\sqrt[4]{\frac{\left[D-\frac{8t_r}{3T}\right]\pi^2\frac{t_r}{T}}{\Psi}}$
7. 	$I_{rms}=I_o\sqrt{\frac{1}{3}}$ $I_{rms}'=\frac{2I_o}{T\sqrt{D(1-D)}}$	$\sum_{n=1}^{\infty}\frac{2I_o\text{Sin}(n\pi D)}{\pi^2n^2D(1-D)}\text{Sin}(n\omega t)$	$\Delta_{opt}=\sqrt[4]{\frac{\pi^2D(1-D)}{3\Psi}}$
8. 	$I_{rms}=I_o\sqrt{\frac{D}{3}}$ $I_{rms}'=\frac{2I_o}{\sqrt{DT}}$	$\frac{I_oD}{2}+\sum_{n=1}^{\infty}\frac{4I_o}{\pi^2n^2D}\text{Sin}^2\left(\frac{n\pi D}{2}\right)\text{Cos}(n\omega t)$	$\Delta_{opt}=\sqrt[4]{\frac{\pi^2D}{3\Psi}}$
9. 	$I_{rms}=I_o\sqrt{\frac{D}{3}}$ $I_{rms}'=\frac{4I_o}{\sqrt{DT}}$	$\sum_{n=1,\,odd}^{\infty}\frac{16I_o}{\pi^2n^2D}\text{Sin}^2\left(\frac{n\pi D}{4}\right)\text{Cos}(n\omega t)$	$\Delta_{opt}=\sqrt[4]{\frac{\pi^2D}{12\Psi}}$

♦ In waveform 2 for $n=k=1/2D\in N$ (the set of natural numbers), and in waveform 3 for $n=k=1/D\in N$, the {expression in curly brackets} is replaced by $\pi^2/16$.