Optimizing the AC Resistance of Multilayer Transformer Windings with Arbitrary Current Waveforms

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***Abstract*—AC losses due to nonsinusoidal current waveforms have been found by calculating the losses at harmonic frequencies when the Fourier coefficients are known. An optimized foil or layer thickness in a winding may be found by applying the Fourier analysis over a range of thickness values. This paper presents a new formula for the optimum foil or layer thickness, without the need for Fourier coefficients and calculations at harmonic frequencies. The new formula requires the rms value of the current waveform and the rms value of its derivative. It is simple, straightforward and applies to any periodic waveshape.**

***Index Terms*—AC resistance, magnetic circuits, optimization, switching circuits, transformers.**

NOMENCLATURE

d	Thickness of foil or layer.
D	Duty cycle.
f	Fundamental frequency of the current waveform in Hz.
I_{dc}	Average value of current.
I_n	RMS value of the nth harmonic.
I_{rms}	RMS value of the current waveform.
I'_{rms}	RMS value of the derivative of the current waveform.
k_{p_n}	Ratio of the ac resistance to dc resistance at nth harmonic frequency.
N	Number of turns per layer.
n	Harmonic number.
p	Number of layers.
r_o	Radius of bare wire in wire-wound winding.
R_{ac}	AC resistance of a winding with sinusoidal excitation.
R_{dc}	DC resistance of a winding.
R_{eff}	Effective ac resistance of a winding, with arbitrary current waveform.
R_δ	DC resistance of a winding of thickness δ_0.
t_r	Rise time (0–100%).
T	Period of the current waveform.
δ_0	$\sqrt{2/\omega\mu_0\sigma}$, skin depth at fundamental frequency, $\omega = 2\pi f$.
δ_n	Skin depth at the nth harmonic frequency.
Δ	d/δ_0.
η	Porosity factor (see Fig. 1).
Ψ	$(5p^2-1)/15$.
μ_0	Permeability of free space, $4\pi \times 10^{-7}$ H/m.

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I. INTRODUCTION

TRANSFORMERS are operated at high frequencies in order to reduce their size [1]. Switching circuits and resonant circuits have greatly improved the efficiencies of power supplies. These power supplies have nonsinusoidal current waveforms and give rise to additional ac losses due to harmonics. AC resistance effects due to sinusoidal currents were treated by Bennett and Larson [2] and this work was tailored specifically for transformers by Dowell [3]. These works are based on a one-dimensional solution of the diffusion equation as applied to conducting parallel plates. Dowell's formula has been found to reliably predict the increased resistance in cylindrical windings where the foil or layer thickness is less than 10% of the radius of curvature. The formula has been utilized in many applications such as planar magnetics by Kassakian [4] and Sullivan [5], matrix transformers by Williams [6], toroidal inductors by Cheng [7], distributed air-gaps by Evans [8] and slot bound conductors by Hanselman [9].

With the advent of switch mode power supplies, attention switched to nonsinusoidal current waveforms. These currents were decomposed into Fourier components; the harmonic components are orthogonal so that the total loss is equal to the sum of the losses calculated by Dowell's formula for the amplitude and frequency of each harmonic in turn. Venkatraman [10] showed that for a pulsed waveform typical of a forward converter, there is an optimum layer thickness to minimize ac losses. Carsten [11] extended the analysis to square waveforms, which are encountered in full bridge converters and to triangular waveforms, which occur in filter chokes. Vandelac [12] extended the analysis to flyback converters. The optimum layer thickness is found as follows.

1) Calculate the Fourier coefficients.
2) Calculate the losses at each harmonic frequency.
3) Calculate the total losses for each thickness in a range of values.
4) Read the optimum thickness from a graph of ac resistance versus layer thickness.

Typically this might involve loss calculations at up to 30 harmonics for up to 10 thickness values in order to find the optimum

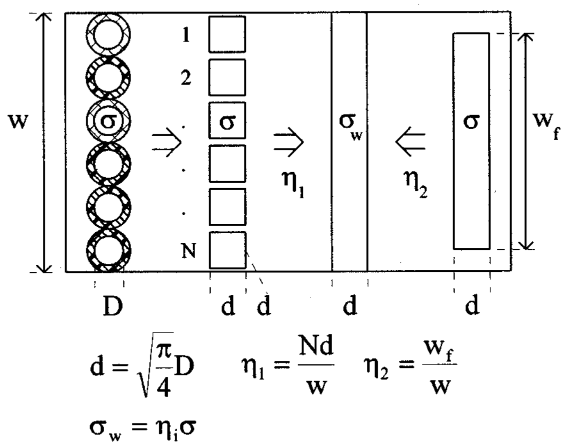


Fig. 1. Porosity factor for foils and round conductors.

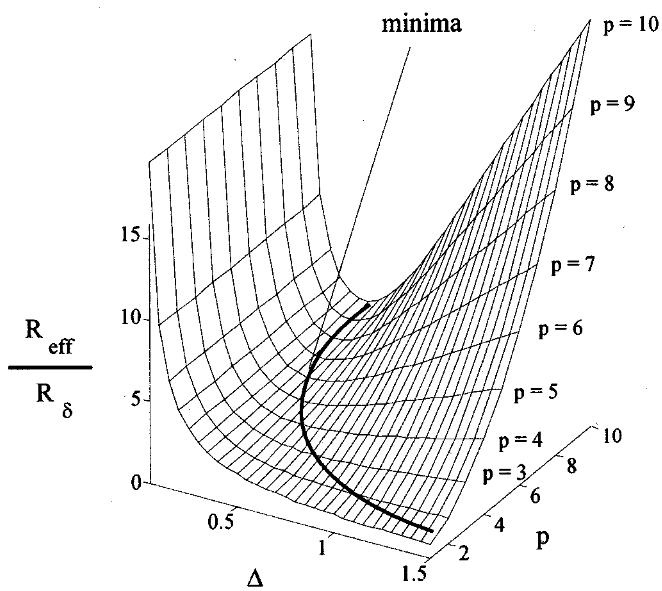


Fig. 2. Plot of ac resistance versus Δ and number of layers p.

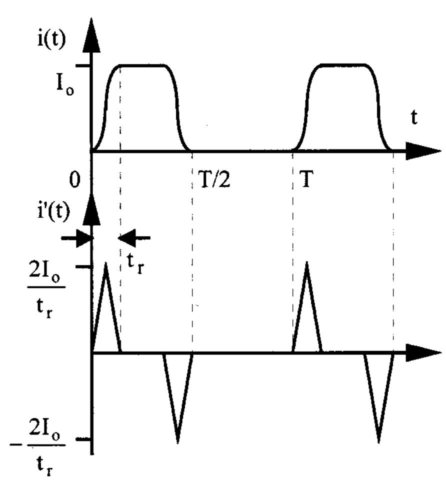


Fig. 3. Pulsed current waveform and its derivative.

value. Furthermore Fourier coefficients are only available for a few waveforms.

TABLE I
VALIDATION RESULTS, $p = 6$, $D = 0.4$, $t_r/T = 4\%$

Waveform No.	Fourier Analysis	New Formula
1	0.539	0.538
2	0.490	0.481
3	0.348	0.340
4	0.429	0.415
5	0.416	0.389
6	0.328	0.314
7	0.515	0.507
8	0.469	0.458
9	0.333	0.324

This paper presents a new formula for ac resistance and the optimum layer thickness for any current waveform. The formula only requires knowledge of the rms value of the current waveform and the rms value of its derivative. Both these quantities can be easily measured or calculated with simulation programs such as PSPICE. The results are just as accurate as the cumbersome method based on Fourier analysis.

II. AC RESISTANCE

The solution of the diffusion equation for cylindrical windings is detailed in the Appendix. The asymptotic expansion of the Bessel functions in the solution leads to Dowell's formula for the ac resistance of a coil with p layers, with sinusoidal excitation. The real part of Dowell's formula gives the ac to dc resistance factor:

$$\frac{R_{\rm ac}}{R_{\rm dc}} = \Delta\left[\frac{\sinh 2\Delta + \sin 2\Delta}{\cosh 2\Delta - \cos 2\Delta} + \frac{2(p^2-1)}{3}\,\frac{\sinh \Delta - \sin \Delta}{\cosh \Delta + \cos \Delta}\right] \quad (1)$$

where Δ is the ratio of the layer thickness d to the skin depth δ_0. This is a very good approximation to the original cylindrical solution, particularly if the layer thickness is less than 10% of the radius of curvature. Windings which consist of round conductors, or foils which do not extend the full winding window, may be treated as foils with equivalent thickness d and effective conductivity $\sigma_w = \eta\sigma$. This calculation is shown graphically in Fig. 1, a detailed treatment of wire conductors is given by Ferreira [13] and Jongsma [14]. The orthogonality of skin and proximity effects in wire windings is described by Ferreira [13].

The trigonometric and hyperbolic functions in (1) may be represented by the series expansions

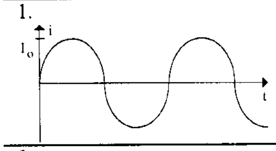
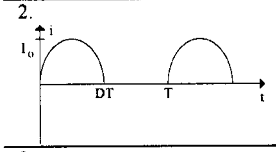
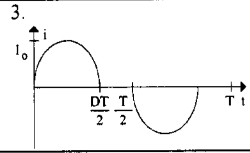
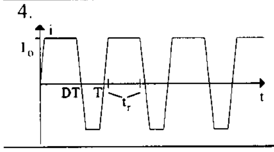
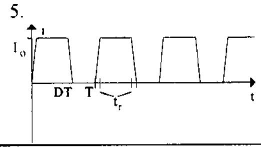
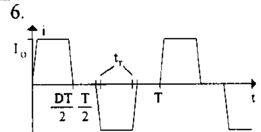
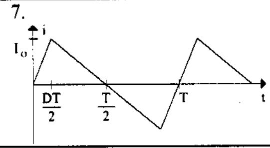
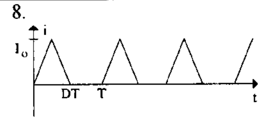
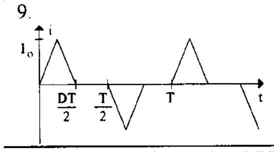
$$\frac{\sinh 2\Delta + \sin 2\Delta}{\cosh 2\Delta - \cos 2\Delta} \approx \frac{1}{\Delta} + \frac{4}{45}\Delta^3 - \frac{16}{4725}\Delta^7 + O(\Delta^{11}) \quad (2)$$

$$\frac{\sinh \Delta - \sin \Delta}{\cosh \Delta + \cos \Delta} \approx \frac{1}{6}\Delta^3 - \frac{17}{2520}\Delta^7 + O(\Delta^{11}). \quad (3)$$

If only terms up to the order of Δ^3 are used, the relative error incurred in (2) is less than 1.2% for $\Delta < 1.2$ and the relative error in (3) is less than 4.1% for $\Delta < 1$ and is less than 8.4%

TABLE II

FORMULAS FOR THE OPTIMUM THICKNESS OF A WINDING FOR VARIOUS WAVEFORMS, $\Psi = (5p^2-1)/15, p =$ NO. OF LAYERS, $\mathrm{sinc}(x) = \sin(x)/x$

Current Waveform	I_{rms} and I_{rms}'	Fourier Series, i(t)	Δ_{opt}
1. 	$I_{rms} = \frac{I_o}{\sqrt{2}}$ $I_{rms}' = \frac{2\pi}{T\sqrt{2}} I_o$	$I_o \mathrm{Sin}(\omega t)$	$\Delta_{opt} = \sqrt[4]{\frac{1}{\Psi}}$
2. 	$I_{rms} = I_o\sqrt{\frac{D}{2}}$ $I_{rms}' = I_o \frac{\pi}{DT}\sqrt{\frac{D}{2}}$	$\frac{2DI_o}{\pi} + \sum_{n=1}^{\infty} \frac{4DI_o}{\pi}\left\{\frac{\mathrm{Cos}(n\pi D)}{(1-4n^2D^2)}\right\}\mathrm{Cos}(n\omega t)$ ♦	$\Delta_{opt} = \sqrt[4]{\frac{4D^2}{\Psi}}$
3. 	$I_{rms} = I_o\sqrt{\frac{D}{2}}$ $I_{rms}' = I_o \frac{2\pi}{DT}\sqrt{\frac{D}{2}}$	$\sum_{n=1,\,odd}^{\infty} \frac{4DI_o}{\pi}\left\{\frac{\mathrm{Cos}(n\pi D/2)}{(1-n^2D^2)}\right\}\mathrm{Cos}(n\omega t)$ ♦	$\Delta_{opt} = \sqrt[4]{\frac{D^2}{\Psi}}$
4. 	$I_{rms} = I_o\sqrt{1-\frac{8t_r}{3T}}$ $I_{rms}' = I_o\sqrt{\frac{4}{t_r T}}$	$I_o(2D-1) + \sum_{n=1}^{\infty}\frac{4I_o}{n\pi}\mathrm{Sin}(n\pi D)$ $\times \mathrm{Sinc}\left(n\pi\frac{2t_r}{T}\right)\mathrm{Cos}(n\omega t)$	$\Delta_{opt} = \sqrt[4]{\frac{\left[1-\frac{8t_r}{3T}\right]\pi^2\frac{t_r}{T}}{\Psi}}$
5. 	$I_{rms} = I_o\sqrt{D-\frac{4t_r}{3T}}$ $I_{rms}' = I_o\sqrt{\frac{2}{t_r T}}$	$I_o\left(D-\frac{t_r}{T}\right) + \sum_{n=1}^{\infty}\frac{2I_o}{n\pi}\mathrm{Sin}\left(n\pi\left(D-\frac{t_r}{T}\right)\right)$ $\times \mathrm{Sinc}\left(n\pi\frac{t_r}{T}\right)\mathrm{Cos}(n\omega t)$	$\Delta_{opt} = \sqrt[4]{\frac{\left[D-\frac{4t_r}{3T}\right]2\pi^2\frac{t_r}{T}}{\Psi}}$
6. 	$I_{rms} = I_o\sqrt{D-\frac{8t_r}{3T}}$ $I_{rms}' = I_o\sqrt{\frac{4}{t_r T}}$	$\sum_{n=1,\,odd}^{\infty}\frac{4I_o}{n\pi}\mathrm{Sin}\left(n\pi\left(\frac{D}{2}-\frac{t_r}{T}\right)\right)$ $\times \mathrm{Sinc}\left(n\pi\frac{t_r}{T}\right)\mathrm{Cos}(n\omega t)$	$\Delta_{opt} = \sqrt[4]{\frac{\left[D-\frac{8t_r}{3T}\right]\pi^2\frac{t_r}{T}}{\Psi}}$
7. 	$I_{rms} = I_o\sqrt{\frac{1}{3}}$ $I_{rms}' = \frac{2I_o}{T\sqrt{D(1-D)}}$	$\sum_{n=1}^{\infty}\frac{2I_o\mathrm{Sin}(n\pi D)}{\pi^2 n^2 D(1-D)}\mathrm{Sin}(n\omega t)$	$\Delta_{opt} = \sqrt[4]{\frac{\pi^2 D(1-D)}{3\Psi}}$
8. 	$I_{rms} = I_o\sqrt{\frac{D}{3}}$ $I_{rms}' = \frac{2I_o}{\sqrt{DT}}$	$\frac{I_o D}{2} + \sum_{n=1}^{\infty}\frac{4I_o}{\pi^2 n^2 D}\mathrm{Sin}^2\left(\frac{n\pi D}{2}\right)\mathrm{Cos}(n\omega t)$	$\Delta_{opt} = \sqrt[4]{\frac{\pi^2 D}{3\Psi}}$
9. 	$I_{rms} = I_o\sqrt{\frac{D}{3}}$ $I_{rms}' = \frac{4I_o}{\sqrt{DT}}$	$\sum_{n=1,\,odd}^{\infty}\frac{16I_o}{\pi^2 n^2 D}\mathrm{Sin}^2\left(\frac{n\pi D}{4}\right)\mathrm{Cos}(n\omega t)$	$\Delta_{opt} = \sqrt[4]{\frac{\pi^2 D}{12\Psi}}$

♦ In waveform 2 for $n = k = 1/2D \in \mathbf{N}$ (the set of natural numbers), and in waveform 3 for $n = k = 1/D \in \mathbf{N}$, the {expression in curly brackets} is replaced by $\pi/4$.

if $\Delta < 1.2$. The asymptotic values of the functions on the left hand side of (2) and (3) are 1 for $\Delta > 2.5$. Terms up to the order of Δ^3 are sufficiently accurate to account for the Fourier harmonics which are used to predict the optimum value of Δ which is normally in the range 0.3–1.

Thus (1) becomes

$$\frac{R_{\mathrm{ac}}}{R_{\mathrm{dc}}} = 1 + \frac{\Psi}{3}\Delta^4 \tag{4}$$

where

$$\Psi = \frac{5p^2-1}{15}. \tag{5}$$

An arbitrary periodic current waveform, may be represented by its Fourier series

$$i(t) = I_{\mathrm{dc}} + \sum_{n=1}^{\infty} a_n \cos n\omega t + b_n \sin n\omega t. \tag{6}$$

The sine and cosine terms may be combined to give an alternative form

$$i(t) = I_{\rm dc} + \sum_{n=1}^{\infty} c_n \cos(n\omega t + \varphi_n) \tag{7}$$

where $I_{\rm dc}$ is the dc value of $i(t)$ and c_n is the amplitude of the nth harmonic with corresponding phase φ_n. The rms value of the nth harmonic is $I_n = c_n/\sqrt{2}$.

The total power loss due to all the harmonics is

$$P = R_{\rm dc} I_{\rm dc}^2 + R_{\rm dc} \sum_{n=1}^{\infty} k_{p_n} I_n^2 \tag{8}$$

where k_{p_n} is the ac resistance factor at the nth harmonic frequency, which may be found from (1)

$$k_{p_n} = \sqrt{n}\Delta \left[\frac{\sinh 2\sqrt{n}\Delta + \sin 2\sqrt{n}\Delta}{\cosh 2\sqrt{n}\Delta - \cos 2\sqrt{n}\Delta} + \frac{2(p^2-1)}{3} \frac{\sinh \sqrt{n}\Delta - \sin \sqrt{n}\Delta}{\cosh \sqrt{n}\Delta + \cos \sqrt{n}\Delta} \right]. \tag{9}$$

$R_{\rm eff}$ is the ac resistance due to $i(t)$ so that $P = R_{\rm eff} I_{\rm rms}^2$, $I_{\rm rms}$ being the rms value of $i(t)$. Thus the ratio of effective ac resistance to dc resistance is

$$\frac{R_{\rm eff}}{R_{\rm dc}} = \frac{I_{\rm dc}^2 + \sum_{n=1}^{\infty} k_{p_n} I_n^2}{I_{\rm rms}^2}. \tag{10}$$

The skin depth at the nth harmonic is $\delta_n = \delta_0/\sqrt{n}$ and, from (4), the ac resistance factor at the nth harmonic frequency is

$$k_{p_n} = 1 + \frac{\Psi}{3} n^2 \Delta^4. \tag{11}$$

Substituting (11) into (10) yields

$$\frac{R_{\rm eff}}{R_{\rm dc}} = \frac{I_{\rm dc}^2 + \sum_{n=1}^{\infty} I_n^2 + \frac{\Psi}{3}\Delta^4 \sum_{n=1}^{\infty} n^2 I_n^2}{I_{\rm rms}^2}. \tag{12}$$

The rms value of the current in terms of its harmonics is

$$I_{\rm rms}^2 = I_{\rm dc}^2 + \sum_{n=1}^{\infty} I_n^2. \tag{13}$$

The derivative of $i(t)$ in (7) is

$$\frac{di}{dt} = -\omega \sum_{n=1}^{\infty} n c_n \sin(n\omega t + \phi_n) \tag{14}$$

and the rms value of the derivative of the current is [15]

$$I_{\rm rms}'^2 = \omega^2 \sum_{n=1}^{\infty} \frac{n^2 c_n^2}{2} = \omega^2 \sum_{n=1}^{\infty} n^2 I_n^2 \tag{15}$$

which, upon substitution into (12) using (13), yields

$$\frac{R_{\rm eff}}{R_{\rm dc}} = 1 + \frac{\Psi}{3}\Delta^4 \left[\frac{I'_{\rm rms}}{\omega I_{\rm rms}}\right]^2. \tag{16}$$

This is a straightforward expression for the effective resistance of a winding with an arbitrary current waveform and it may be evaluated without knowledge of the Fourier coefficients of the waveform.

III. The Optimum Conditions

There is an optimum value of d, which gives a minimum value of effective ac resistance. Define R_δ as the dc resistance of a foil of thickness δ_0 such that

$$\frac{R_\delta}{R_{\rm dc}} = \frac{d}{\delta_0} = \Delta \tag{17}$$

which implies that

$$\frac{R_{\rm eff}}{R_{\rm dc}} = \Delta \frac{R_{\rm eff}}{R_\delta}. \tag{18}$$

Evidently, a plot of $R_{\rm eff}/R_\delta$ versus Δ has the same shape as a plot of $R_{\rm eff}$ versus d at a given frequency. A 3-D plot of $R_{\rm eff}/R_\delta$ versus Δ with p, the number of layers in the winding, on the third axis is shown in Fig. 2.

For each value of p there is an optimum value of Δ where the ac resistance of the winding is minimum. These optimum points lie on the line marked minima in Fig. 2 and the corresponding value of the optimum layer thickness is

$$d_{opt} = \Delta_{opt}\delta_0. \tag{19}$$

From (18), using (16)

$$\frac{R_{\rm eff}}{R_\delta} = \frac{1}{\Delta} + \frac{\Psi}{3}\Delta^3 \left[\frac{I'_{\rm rms}}{\omega I_{\rm rms}}\right]^2. \tag{20}$$

The optimum value of Δ is found by taking the derivative of (20) and setting it to zero

$$\frac{d}{d\Delta}\left(\frac{R_{\rm eff}}{R_\delta}\right) = -\frac{1}{\Delta^2} + \Psi\Delta^2 \left[\frac{I'_{\rm rms}}{\omega I_{\rm rms}}\right]^2 = 0. \tag{21}$$

The optimum value of Δ is

$$\Delta_{opt} = \frac{1}{\sqrt[4]{\Psi}} \sqrt{\frac{\omega I_{\rm rms}}{I'_{\rm rms}}}. \tag{22}$$

Substituting this result into (16) yields the optimum value of the effective ac resistance with an arbitrary periodic current waveform:

$$\left(\frac{R_{\rm eff}}{R_{\rm dc}}\right)_{opt} = \frac{4}{3}. \tag{23}$$

Jongsma [14] and Snelling [16] have already established this result for sinusoidal excitation. The corresponding value for wire conductors with sinusoidal excitation is 3/2 [14], [16].

We may also write (16) in term of Δ_{opt}

$$\frac{R_{\rm eff}}{R_{\rm dc}} = 1 + \frac{1}{3}\left(\frac{\Delta}{\Delta_{opt}}\right)^4. \tag{24}$$

We now have a set of simple formulas with which to find the optimum value of the foil or layer thickness of a winding and its effective ac resistance, these formulas are based on the rms value of the current waveform and the rms value of its derivative.

IV. Validation

Consider the pulsed current waveform in Fig. 3 along with its derivative, which is typical of a forward converter.

This waveform has a Fourier series:

$$i(t) = I_o\left(\frac{1}{2} - \frac{t_r}{T}\right) + \sum_{n=1}^{\infty} \frac{4I_o}{n^3\pi^3\left(\frac{t_r}{T}\right)^2} \cdot \left[1 - \cos\left(\frac{n\pi t_r}{T}\right)\right] \sin\left(n\pi\left(\frac{1}{2} - \frac{t_r}{T}\right)\right) \cdot \cos\left(n\left(\omega t - \frac{\pi}{2}\right)\right). \tag{25}$$

The rms value of $i(t)$ and the the rms value of its derivative are

$$I_{\rm rms} = I_o\sqrt{0.5 - \frac{37t_r}{30T}}$$
$$I'_{\rm rms} = I_o\sqrt{\frac{8}{3t_rT}}. \tag{26}$$

The optimum value of Δ given by the Fourier series (25) along with (9), (10), and (18), for $p = 6$ and $t_r/T = 4\%$ is 0.418 and the value given by the proposed formula (22) is 0.387 which represents an error of 7.4%.

Waveform 5 in Table II is an approximation to the pulse in Fig. 3 and the optimum value of Δ using Fourier analysis is 0.448 which represents an error of 7.2% when compared to the Fourier analysis of the waveform given by (25). Evidently waveforms with known Fourier series are often approximations to the actual waveform and can give rise to errors which are of the same order as the new formula, which is simpler to evaluate.

At 50 kHz the skin depth in copper is 0.295 mm. With $\Delta_{opt} = 0.418$, $d_{opt} = 0.295 \times 0.418 = 0.123$ mm.

The new formula may be validated by comparing the value of Δ_{opt} obtained with (22) and the value obtained with Fourier analysis by plotting $R_{\rm eff}/R_\delta$, using (18) along with (9) and (10), over a range of values of Δ and finding the optimum value. The results are shown in Table I for the waveforms in Table II. In general the agreement is within 3%, with the exception of waveform 5 where the error is 6.5%. For the Fourier analysis 19 harmonics were evaluated and $R_{\rm eff}/R_\delta$ was calculated for 20 values of Δ. This means that (9) was computed 380 times for each waveform in order to find the optimum layer thickness, (22) was computed once for the same result. For 1–3 layers the accuracy of the proposed formula is not very good, however, as evidenced by Fig. 2, the plot of $R_{\rm eff}/R_\delta$ is almost flat in the region of the optimum value of Δ, and therefore the error in the ac resistance is negligible.

V. Conclusions

A new formula has been presented to find the optimum foil or layer thickness in a multilayer winding. The formula applies to any arbitrary periodic current waveform. It is computationally easier to use than Fourier analysis while enjoying the same level of accuracy. It has a wider range of application than the Fourier approach by virtue of its simplicity.

Appendix
AC Resistance in a Cylindrical Conductor

For a magnetoquasistatic system, Maxwell's equations in a linear homogeneous isotropic medium take the following form:

$$\nabla \times \boldsymbol{H} = \sigma\boldsymbol{E}, \tag{A1}$$
$$\nabla \times \boldsymbol{E} = -\mu_0\frac{\partial \boldsymbol{H}}{\partial t}. \tag{A2}$$

The annular cylindrical conducting layer, shown in Fig. 4 carries a sinusoidal current $i_\phi(t) = I_\phi e^{j\omega t}$. The conductivity of the conducting medium is σ and the physical dimensions are shown in Fig. 4. H_- and H_+ are the magnetic fields parallel to the inside and outside surfaces of the cylinder, respectively. We shall see shortly that H_- and H_+ are independent of z.

Assuming cylindrical symmetry, the various components of the electric field intensity $\boldsymbol{E}$ and the magnetic field intensity $\boldsymbol{H}$ inside the cylinder, in cylindrical coordinates (r, ϕ, z), satisfy the following identities:

$$E_r = 0, \qquad E_z = 0, \qquad \frac{\partial E_\phi}{\partial z} = 0, \tag{A3}$$
$$H_r = 0, \qquad H_\phi = 0, \qquad \frac{\partial H_z}{\partial \phi} = 0. \tag{A4}$$

The two Maxwell's equations above then reduce to

$$-\frac{\partial H_z}{\partial r} = \sigma E_\phi, \tag{A5}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi) = -j\omega\mu_0 H_z. \tag{A6}$$

Since $\boldsymbol{H}$ has only a z-component and $\boldsymbol{E}$ has only a ϕ-component, we drop the subscripts without ambiguity. Furthermore, the electric and magnetic field intensities are divergence-free and so it follows that E and H are functions of r only. Substituting the expression for E given by (A5) into (A6) then yields the ordinary differential equation

$$\frac{d^2H}{dr^2} + \frac{1}{r}\frac{dH}{dr} - j\omega\mu_0\sigma H = 0. \tag{A7}$$

This is a *modified Bessel's equation*. The general solution is

$$H(r) = AI_0(mr) + BK_0(mr) \tag{A8}$$

where I_0 and K_0 are modified Bessel functions of the first and second kind, of order 0, and $m = \sqrt{j\omega\mu_0\sigma}$, so that the argument of I_0 and K_0 is complex. We take the principal value of the square root, that is $\sqrt{j} = e^{j\pi/4}$. The coefficients A and B are determined from the boundary conditions and will be complex. It is worth noting that the solutions of (A7) are in fact combinations of the Kelvin functions with real argument *viz.* $\mathrm{ber}(r\sqrt{\omega\mu_0\sigma})$, $\mathrm{bei}(r\sqrt{\omega\mu_0\sigma})$, $\mathrm{ker}(r\sqrt{\omega\mu_0\sigma})$ and $\mathrm{kei}(r\sqrt{\omega\mu_0\sigma})$, though in our analysis we find it more convenient to use the modified Bessel functions with complex argument.

A typical transformer cross-section is shown in Fig. 5(a) with associated M.M.F. diagram and current density distribution for a two-turn primary and a three-turn secondary winding. The physical dimensions of a generalized nth layer are shown in Fig. 5(b) (the innermost layer is counted as layer 1). We assume that the magnetic material in the core is ideal ($\mu_r \to \infty$, $\sigma \to 0$) so that the magnetic field intensity goes to zero inside the core. We also assume that the dimension w is much greater than the radial dimensions so that end effects are taken as negligible.

Invoking Ampere's law for the closed loops C_1 and C_2 in a high permeability core ($\mu_r \gg 1$):

$$H_o = \frac{NI}{w} \tag{A9}$$

where N is the number of turns in layer n, each carrying constant current I. n here refers to the layer number and should not be confused with the harmonic number n as used in the body of the paper. Applying the inner and outer boundary conditions for

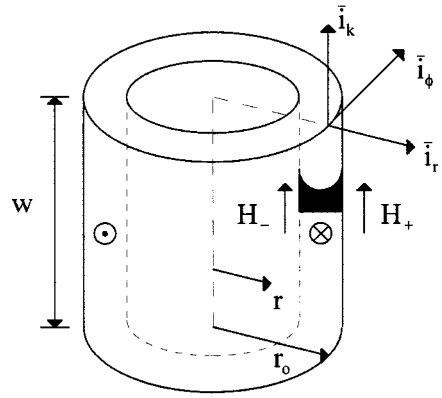


Fig. 4. Conducting cylinder.

layer n, i.e., $H(r_{n_i}) = (n-1)H_o$ and $H(r_{n_o}) = nH_o$ to the general solution (A8) we obtain the coefficients

$$A = \frac{[(nK_0(mr_{n_i}) - (n-1)K_0(mr_{n_o})]H_o}{I_0(mr_{n_o})K_0(mr_{n_i}) - K_0(mr_{n_o})I_0(mr_{n_i})} \quad \text{(A10)}$$

$$B = \frac{[(-nI_0(mr_{n_i}) +, (n-1)I_0(mr_{n_o})]H_o}{I_0(mr_{n_o})K_0(mr_{n_i}) - K_0(mr_{n_o})I_0(mr_{n_i})}. \quad \text{(A11)}$$

The corresponding value of $E(r)$ is found from (A5), that is

$$E(r) = -\frac{1}{\sigma}\,\frac{dH(r)}{dr}. \quad \text{(A12)}$$

Using the modified Bessel function identities

$$\frac{d}{dr}\,I_0(mr) = mI_1(mr) \quad \text{(A13)}$$

$$\frac{d}{dr}\,K_0(mr) = -mK_1(mr) \quad \text{(A14)}$$

the electric field intensity is then given by

$$E(r) = -\frac{m}{\sigma}[AI_1(mr) - BK_1(mr)]. \quad \text{(A15)}$$

The Poynting vector $\boldsymbol{E} \times \boldsymbol{H}$ [17] represents the energy flux density per unit area crossing the surface per unit time. In the cylindrical coordinate system illustrated in Fig. 4, the power per unit area into the cylinder is given by $\boldsymbol{E} \times \boldsymbol{H}$ on the inside surface and $-\boldsymbol{E} \times \boldsymbol{H}$ on the outside surface. Since $\boldsymbol{E}$ and $\boldsymbol{H}$ are orthogonal, the magnitude of the Poynting vector is simply the product $E(r)H(r)$ and its direction is radially outwards.

The power per unit length (around the core) of the inside surface of layer n is

$$\begin{aligned} P_{n_i} &= E(r_{n_i})H(r_{n_i})w \\ &= -\frac{m}{\sigma}(n-1)H_o w[AI_1(mr_{n_i}) - BK_1(mr_{n_i})]. \end{aligned} \quad \text{(A16)}$$

A and B are given by (A10) and (A11), respectively, H_o is given by (A9) so

$$\begin{aligned} P_{n_i} = \frac{N^2I^2m}{\sigma w \Psi}\{&(n-1)^2[I_0(mr_{n_o})K_1(mr_{n_i}) \\ &+ K_0(mr_{n_o})I_1(mr_{ni})] - n(n-1) \\ &\cdot [I_0(mr_{n_i})K_1(mr_{n_i}) + K_0(mr_{n_i})I_1(mr_{n_i})]\} \end{aligned} \quad \text{(A17)}$$

where we define

$$\Psi = I_0(mr_{n_o})K_0(mr_{n_i}) - K_0(mr_{n_o})I_0(mr_{n_i}). \quad \text{(A18)}$$

In a similar fashion, we find the power per unit length (around the core) of the outside surface of layer n is

$$\begin{aligned} P_{n_o} &= -E(r_{n_o})H(r_{n_o})w \\ &= \frac{m}{\sigma}\,nH_o w[AI_1(mr_{n_o}) - BK_1(mr_{n_o})] \\ &= \frac{N^2I^2m}{\sigma w \Psi}\,\{n^2[I_0(mr_{n_i})K_1(mr_{n_o}) \\ &\quad + K_0(mr_{n_i})I_1(mr_{n_o})] - n(n-1) \\ &\quad \cdot [I_0(mr_{n_o})K_1(mr_{n_o}) + K_0(mr_{n_o})I_1(mr_{n_o})]\}. \end{aligned} \quad \text{(A19)}$$

The minus sign is required to find the power into the outer surface.

We now assume that $mr \gg 1$ and use the leading terms in the asymptotic approximations for the modified Bessel functions in (A17) and (A19) [noting for purposes of validity $\arg(mr) = \pi/4 < \pi/2$]:

$$\begin{aligned} I_0(mr) &\approx \frac{1}{\sqrt{2\pi mr}}\,e^{mr}, \quad I_1(mr) \approx \frac{1}{\sqrt{2\pi mr}}\,e^{mr}, \\ K_0(mr) &\approx \sqrt{\frac{\pi}{2mr}}\,e^{-mr}, \quad K_1(mr) \approx \sqrt{\frac{\pi}{2mr}}\,e^{-mr}. \end{aligned} \quad \text{(A20)}$$

Substituting these into (A17) and (A19) and rearranging, yields the total power dissipation for layer n as

$$\begin{aligned} P_{n_i} + P_{n_o} = \frac{N^2I^2m}{\sigma w}\Bigg[&(2n^2 - 2n + 1)\,\coth(md_n) \\ &- \frac{n^2 - n}{\sinh(md_n)}\left(\sqrt{\frac{r_{n_o}}{r_{n_i}}} + \sqrt{\frac{r_{n_i}}{r_{n_o}}}\right)\Bigg] \end{aligned} \quad \text{(A21)}$$

where $d_n \equiv r_{n_o} - r_{n_i}$ is the thickness of layer n. This result was obtained from the Poynting vector for the complex field intensities, so the real part represents the actual power dissipation.

We now assume that each layer has constant thickness d, so that $d_n = d$ (independent of n). Furthermore we assume that $d \ll r_{n_i}$. Then using the Taylor expansion

$$\sqrt{1+\varepsilon} + \frac{1}{\sqrt{1+\varepsilon}} = 2 + \frac{\varepsilon^2}{4} + O(\varepsilon^3) \quad \text{(A22)}$$

it follows that if $d/r_{n_i} < 10\%$, the error incurred by approximating the sum of the square roots in (A21) by 2 is in the order of 0.1%. Then the total power dissipation in layer n becomes

$$\begin{aligned} &\Re(P_{n_i} + P_{n_o}) \\ &\quad \approx \Re\left(\frac{N^2I^2m}{\sigma w}\left[(2n^2 - 2n + 1)\coth(md) - \frac{2(n^2 - n)}{\sinh(md)}\right]\right) \\ &\quad = \Re\left(\frac{N^2I^2m}{\sigma w}\left[\coth(md) + 2(n^2 - n)\tanh\left(\frac{md}{2}\right)\right]\right). \end{aligned} \quad \text{(A23)}$$

The dc power per unit length of layer n is

$$P_{\mathrm{dc}} = R_{\mathrm{dc}}(NI)^2 = \frac{(NI)^2}{\sigma w d}. \quad \text{(A24)}$$

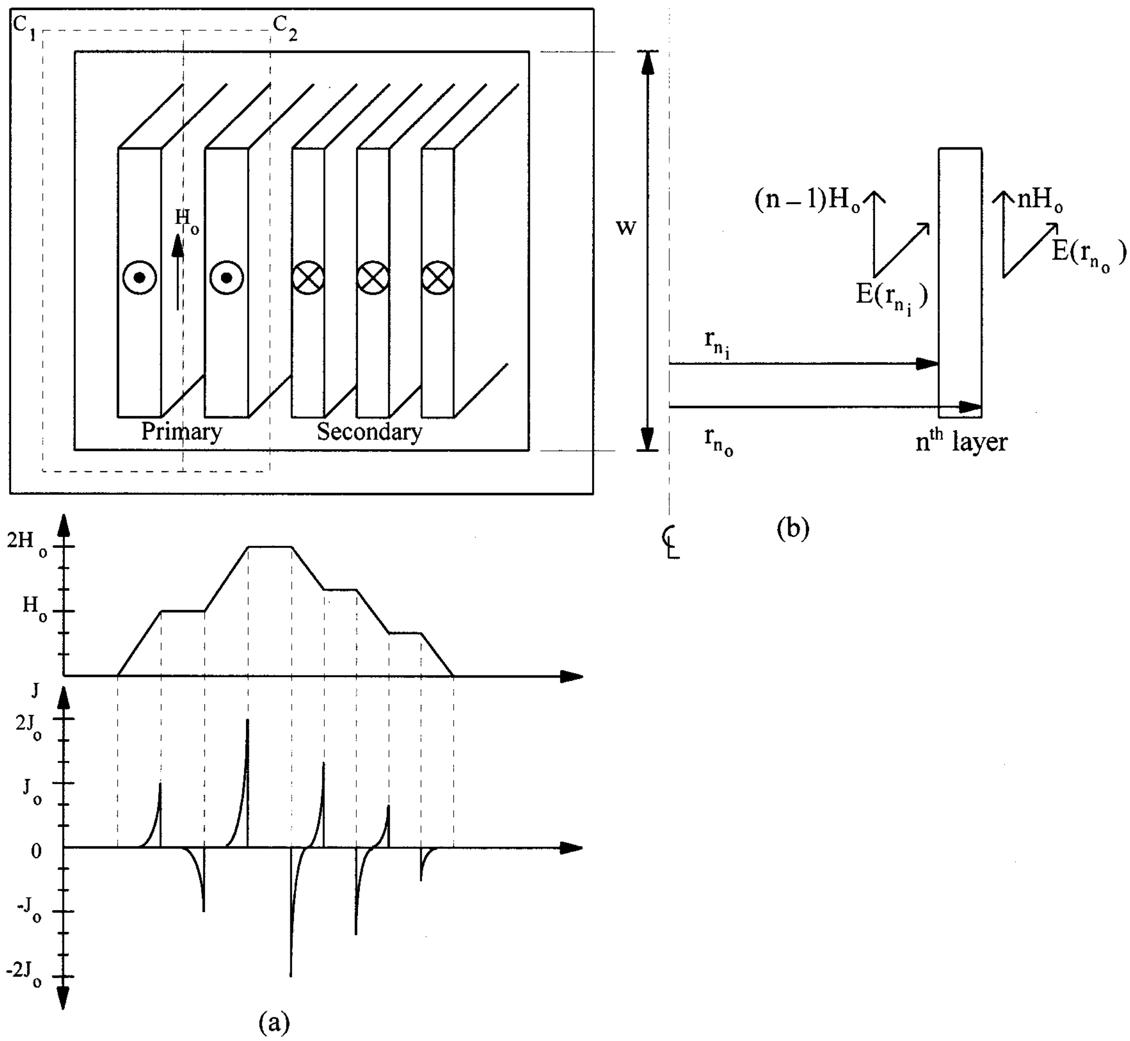


Fig. 5. (a) Transformer cross-section with associated MMF diagram and current density at high frequency and (b) generalized nth layer.

The ac resistance factor is the ratio of the ac resistance to the dc resistance

$$\frac{R_{\mathrm{ac}}}{R_{\mathrm{dc}}} = \frac{\Re(P_{n_i}+P_{n_o})}{P_{\mathrm{dc}}} = \Re\left(md\left[\coth(md)+2(n^2-n)\tanh\left(\frac{md}{2}\right)\right]\right). \tag{A25}$$

Finally the general result for p layers is

$$\frac{R_{\mathrm{ac}}}{R_{\mathrm{dc}}} = \frac{\Re\sum_{n=1}^{p}(P_{n_i}+P_{n_o})}{\sum_{n=1}^{p}P_{\mathrm{dc}}} = \frac{1}{p}\Re\left(md\sum_{n=1}^{p}\left[\coth(md)+2(n^2-n)\tanh\left(\frac{md}{2}\right)\right]\right) = \Re\left(md\left[\coth(md)+\frac{2(p^2-1)}{3}\tanh\left(\frac{md}{2}\right)\right]\right). \tag{A26}$$

From the definition of m

$$md = \sqrt{\omega\mu_0\sigma}\left(\frac{1+j}{\sqrt{2}}\right)d = (1+j)\Delta \tag{A27}$$

where $\Delta = d/\delta_0$ and δ_0 is the skin depth. The ac resistance factor for p layers is then

$$\frac{R_{\mathrm{ac}}}{R_{\mathrm{dc}}} = \Re\left(\Delta(1+j)\left[\coth(\Delta(1+j)) + \frac{2p^2-1}{3}\tanh\left(\frac{\Delta}{2}(1+j)\right)\right]\right) = \Delta\left[\frac{\sinh 2\Delta+\sin 2\Delta}{\cosh 2\Delta-\cos 2\Delta} + \frac{2(p^2-1)}{3}\frac{\sinh\Delta-\sin\Delta}{\cosh\Delta+\cos\Delta}\right]. \tag{A28}$$

This is Dowell's formula.

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References

[1] W. G. Hurley, W. Wölfle, and J. G. Breslin, "Optimized transformer design: Inclusive of high frequency effects," *IEEE Trans. Power Electron.*, vol. 13, pp. 651–659, July 1998.

[2] E. Bennett and S. C. Larson, "Effective resistance to alternating currents of multilayer windings," *Trans. AIEE*, vol. 59, pp. 1010–1016, 1940.

[3] P. L. Dowell, "Effect of eddy currents in transformer windings," *IEE Proc.*, vol. 113, no. 8, pp. 1387–1394, Aug. 1966.

[4] J. G. Kassakian and M. F. Schlecht, "High-frequency high-density converters for distributed power-supply systems," *Proc. IEEE*, vol. 76, pp. 362–376, Apr. 1988.

[5] C. R. Sullivan and S. R. Sanders, "Design of microfabricated transformers and inductors for high-frequency power conversion," *IEEE Trans. Power Electron.*, vol. 11, pp. 228–238, Mar. 1996.

[6] R. Williams, D. A. Grant, and J. Gowar, "Multielement transformers for switched-mode power-supplies: Toroidal designs," *IEE Proc.*, pt. B, vol. 140, no. 2, pp. 152–160, Mar. 1993.

[7] K. W. E. Cheng and P. D. Evans, "Calculation of winding losses in high-frequency toroidal inductors using multistrand conductors," *IEE Proc.—Electron. Power Applicat.*, vol. 142, pp. 313–222, Sept. 1995.

[8] P. D. Evans and W. M. Chew, "Reduction of proximity losses in coupled inductors," *Proc. Inst. Elect. Eng. B*, vol. 138, no. 2, pp. 51–58, Mar. 1991.

[9] D. C. Hanselman and W. H. Peake, "Eddy-current effects in slot-bound conductors," *Proc. Inst. Elect. Eng. B—Electr. Power Appl.*, vol. 142, pp. 131–136, Mar. 1995.

[10] P. S. Venkatraman, "Winding eddy current losses in switch mode power transformers due to rectangular wave currents," in *Proc. Powercon 11*, sec. Section A-1, 1984, pp. 1–11.

[11] B. Carsten, "High frequency conductor losses in switchmode magnetics," *Proc. HPFC*, pp. 155–176, May 1986.

[12] J. Vandelac and P. D. Ziogas, "A novel approach for minimizing high-frequency transformer copper losses," *IEEE Trans. Power Electron.*, vol. 3, no. 3, pp. 166–176, July 1988.

[13] J. A. Ferreira, "Improved analytical modeling of conductive losses in magnetic components," *IEEE Trans. Power Electron.*, vol. 9, no. 1, pp. 127–131, Jan. 1994.

[14] J. Jongsma, "Minimum loss transformer windings for ultrasonic frequencies—Part 3: Transformer winding design," *Phillips Electron. Applicat. Bull.*, vol. E.A.B., no. 35, pp. 211–226, 1978.

[15] C. R. Sullivan, "Optimal choice for number of strands in a Litz-wire transformer winding," in *Proc. 28th Annu. IEEE Power Electron. Spec. Conf.*, 1997, pp. 28–35.

[16] E. C. Snelling, *Soft Ferrites, Properties Applicat.*, 2nd ed. London, U.K.: Butterworths, 1988.

[17] J. A. Ferreira, *Electromagnetic Modeling of Power Electronic Converters*. Boston, MA: Kluwer, 1989.

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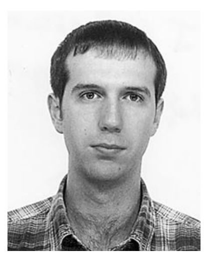

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